

The Monopole Equations in Topological Yang-Mills*

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ABSTRACT

We twist the monopole equations of Seiberg and Witten and show how these equations are realized in topological Yang-Mills theory. A Floer derivative and a Morse functional are found and are used to construct a unitary transformation between the usual Floer cohomologies and those of the monopole equations. Furthermore, these equations are seen to reside in the vanishing self-dual curvature condition of an $OSp(1|2)$ -bundle. Alternatively, they may be seen arising directly from a vanishing self-dual curvature condition on an $SU(2)$ -bundle in which the fermions are realized as spanning the tangent space for a specific background.

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1 Introduction

In this note, we will demonstrate how the monopole equations of ref. [1] for an abelian connection A and $SU(2)$ doublet fermions arise in topological Yang-Mills gauge (TYM) theory [2] and in Floer theory [3] in particular. As we will see the process is remarkably simple. Along the way, we will develop a quantum mechanical system whose ground states have support on the fields which satisfy the twisted monopole equations. What is more, we will find that the inner products of representatives of the cohomology groups so constructed, are formally equal to the Donaldson invariants [4].

Consider the twisted version of the monopole equations. Let \mathcal{T} denote the twisting map, S^\pm the right/left spin bundles over a four dimensional manifold, X , Λ^p the bundle of p -forms and Λ_O^p the bundle of p -forms with odd Grassmann parity. Then for a spinor which in addition to being a section of S^+ is also a doublet of a rigid $SU(2)$ (denoted by \mathcal{S}) so that $M = \sigma(S^+ \otimes \mathcal{S})$, we have $\mathcal{T} : \sigma(S^+ \otimes \mathcal{S}) \rightarrow \Lambda_O^1$; likewise¹, $\mathcal{T} : \sigma(S^- \otimes \mathcal{S}) \rightarrow \Lambda_O^0 \oplus \Lambda_O^{2+}$. With this as background, the equations we are interested in are

$$P_+(D\psi) = 0 \quad , \quad D^*\psi = 0 \quad , \quad F^+ = iP_+(\bar{\psi} \wedge \psi) \quad , \quad (1)$$

where $\psi = \sigma(\Lambda_O^1 \otimes L)$, D is the covariant exterior derivative given by the connection A on L , $P_+ = \frac{1}{2}(I + *)$ is the self-dual projector and the “bar” denotes complex conjugation. In the notation of ref. [1], the elliptic complex on which these twisted monopole equations are realized form the exact sequence

$$0 \longrightarrow \Lambda^0 \xrightarrow{s} \Lambda^1 \oplus (\Lambda_O^1 \otimes L) \xrightarrow{t} \Lambda^{2+} \oplus (\Lambda_O^0 \oplus \Lambda_O^{2+}) \longrightarrow 0 \quad . \quad (2)$$

This complex defines the arena in which we will work in this paper. These equations stand on their own irrespective of our discussion in the previous paragraph; in particular, the value of $w_2(X)$. However, they may be viewed as arising from the twisting of the N=2 versions of the monopole equations on spin manifolds. It should be noted that, unlike TYM, the first equation cannot be obtained by smoothly varying the connection in the third equation.

In the next section we will see how the twisted monopole equations (1) arise by an reduction of a class of zero-action solutions of TYM to a $U(1)$ subgroup of the gauge group. Our intention is not to perform a duality transformation on or any addition (such as twisted N=2 hypermultiplets) to TYM as we would like to see these equations directly in the field space of the quantum field theory for the Donaldson invariants. In this way, we hope to be able to shed light on the connection between the Seiberg-Witten invariants [1, 5, 6, 7] and those of Donaldson [4]. A step in this direction will be made in section 3. Some other directions spawned by this approach will be discussed in section 4. Our conclusions may be found in section 5.

2 In Topological Yang-Mills

Let us focus on obtaining the twisted monopole equations as a special minimum action condition in topological Yang-Mills theory (TYM) on a G -bundle. For simplicity, we will take the structure group to be $SU(2)$.

¹ $P_\pm(\Lambda^2) = \Lambda^{2\pm}$ is the projection to (anti-)self-dual two-forms.

First, we realize that the eqns. (1) cannot arise by breaking the gauge group to $U(1)$. Although breaking the gauge group in TYM is possible by adding²

$$S_m \equiv \{Q, \int_X \text{Tr}(\psi \wedge^* D\phi)\} + \int_X V(\phi) \ , \quad (3)$$

to the action for TYM, the equations cannot be obtained in the low-energy effective theory as all fields are in the adjoint representation here. It is presumably possible to add matter to the TYM theory so that the fermions which appear on the right-hand-side (via a current-current coupling in the effective action) of eqn. (1) were not in the adjoint representation of the gauge group. However, a negative feature of that approach would be to take us outside of the field space of TYM, thus making it difficult to realize the connection with the Donaldson invariants. Thus, we now resort to the explicit breaking of the $SU(2)$ gauge group.

As an ansatz, let us look for field configurations for which the TYM action [2] vanishes. Write the $SU(2)$ generators as $J^a \equiv (J, \bar{J}, J^3)$, similarly³ for the sections of $ad(G)$. Take the following fields to vanish: $\mathcal{A}, \bar{\mathcal{A}}, \lambda, \bar{\lambda}, \phi^3, \chi^3, \eta^3$. We will in addition set $(\phi, \bar{\phi}) = (\nu, \bar{\nu})$, where for now ν is a complex constant. Note that the BRST transformations in this field-restricted sector are $[Q', A] = \psi^3$ and $\{Q', \psi^a\} = 0$.

After integrating out λ^3 , the TYM action becomes

$$S = \int_X \left[\frac{1}{8} F^{3+} \wedge^* F^{3+} + \frac{1}{8|\nu|^2} (\psi \wedge \bar{\psi}) \wedge^* (\psi \wedge \bar{\psi}) - \bar{\chi} \wedge^* D\psi - \chi \wedge^* D\bar{\psi} + \bar{\eta} D^* \psi - \eta D^* \bar{\psi} \right] . \quad (4)$$

The first line in this expression may be written as

$$S_0 = \frac{1}{8} \int_X \left| F^{3+} + \frac{1}{\nu} P_+(\bar{\psi} \wedge \psi) \right|^2 , \quad (5)$$

so long as ν is pure imaginary. Then, invoking the χ, η -equations of motion, we see that the action is zero on non-trivial field equations which satisfy the twisted monopole equations (1) with $\psi \rightarrow |\nu|^{-\frac{1}{2}} \psi$.

Now that we have obtained the twisted monopole equations in Donaldson field space, we might impose them as semi-classical conditions on the polynomial invariants. Indeed, the parameter $|\nu|$ is best defined by the Donaldson invariant

$$\langle W_0 \rangle = \langle \frac{1}{2} \text{Tr}(\phi^2) \rangle = \frac{1}{4} |\nu|^2 , \quad (6)$$

where the brackets mean the evaluation on these special field configurations. For the map from $H_2(X)$ to $H^2(\mathcal{M})$, the observable, $\int_\Sigma W_2$, becomes

$$\langle \int_\Sigma W_2 \rangle = \langle \int_\Sigma \text{Tr}(\frac{1}{2} \psi \wedge \psi + \phi F) \rangle = -2\pi |\nu| c_1(L)[\Sigma] , \quad (7)$$

proportional to the first Chern class of the line bundle.

²Note that $V(\phi)$ is a Higgs potential for the BRST singlet field, which is bounded from below at zero. Although this explicitly introduces a metric (in a volume preserving, diffeomorphism invariant way), as long as V is zero in the low-energy effective field theory, we can ignore this effect.

³We will use the symbol \mathcal{A}^a , with gauge index a , to denote the $SU(2)$ connection; while we will use $A = \mathcal{A}^3$ for the $U(1)$ connection, as in the previous section.

Having recovered the twisted monopole equations by hand from TYM, we now wonder why they should exist in the latter theory in the first place. Well, this is where the “by-hand” procedure we have just performed actually teaches us something. The first and third equations in (1) are nothing but the anti-self-dual condition in disguise. To see this, consider starting off with the equation $\mathcal{F}^+(\mathcal{A}) = 0$ for a $SU(2)$ curvature with connection \mathcal{A} . Then write the connection as a particular background (A) plus a fluctuation ($\hat{\psi}$) via the expressions $\mathcal{A}^3 = A^3$, $\frac{1}{\sqrt{2}}(\mathcal{A}^1 + i\mathcal{A}^2) = \hat{\psi}$ and $\frac{1}{\sqrt{2}}(\mathcal{A}^1 - i\mathcal{A}^2) = \hat{\bar{\psi}}$; *i.e.*, A^3 does not fluctuate while the other connection components do not have background parts. Upon substituting these expressions into the $\mathcal{F}^+ = 0$ equation, dropping the hats on the ψ 's and changing their Grassmann parity we arrive at the first and third twisted monopole equations. Thus we see that it is not surprising that we obtained them in TYM.

3 In Floer Cohomology

We have seen how the twisted monopole equations appear as a minimum action configuration in TYM. We will now identify the analogous Floer cohomology condition. Then we will see that a unitary transformation exists which relates the Floer and monopole cohomologies so constructed. It is important that we will be working in the phase space of the Floer theory. As a point of reference recall that given a closed, orientable 3-manifold, Y , the Floer cohomology operator is (t is an arbitrary real parameter)

$$Q_t = \int_Y \psi_i^a(x) \left(\frac{\delta}{\delta \mathcal{A}_i^a(x)} + \frac{1}{2} t \epsilon^{ijk} \mathcal{F}_{jk a}(x) \right) , \quad (8)$$

for which the representatives of the cohomology groups are the wavefunctionals, $\Psi[\mathcal{A}_i^a, \psi_i^a]$ which satisfy the condition $Q_t \Psi = Q_t^\dagger \Psi = 0$, where $\psi^\dagger = \bar{\chi}$.

As before, let $\mathcal{A} \equiv (\mathcal{A}^\pm, A)$ be the connection on the $SU(2)$ bundle, G , over Y and take ψ^a to be the components of a section of the bundle $(\Lambda_O^1 \otimes G)$. Choose the Morse function to be (see also ref. [8])

$$W'[A, \psi] = \frac{1}{4\pi} \int_Y \left[A \wedge dA + i 2 \bar{\psi} \wedge D(A) \psi \right] , \quad (9)$$

and based on $Q = \int_Y \psi_i^a \frac{\delta}{\delta \mathcal{A}_i^a}$ define the exterior derivative

$$\begin{aligned} Q'_t &= e^{-2\pi t W'[A, \psi]} Q e^{2\pi t W'[A, \psi]} \\ &= \int_Y \left[\psi_i^1 \frac{\delta}{\delta \mathcal{A}_i^1} + \psi_i^2 \frac{\delta}{\delta \mathcal{A}_i^2} \right. \\ &\quad \left. + \psi_i^3 \left(\frac{\delta}{\delta A_i} + \frac{1}{2} t \epsilon^{ijk} F_{jk}(A) - 2it \epsilon^{ijk} (\bar{\psi}_j(x) \psi_k(x)) \right) \right] . \end{aligned} \quad (10)$$

Clearly, a solution of $Q'_t \Psi'[\mathcal{A}_i^a, \psi_i^a] = 0$ is any $\Psi'[A, \psi]$ which has support only on equation (1) written on $X = Y \times \mathbb{R}$,

$$F_{0i}^+ = \epsilon_{ijk} \bar{\psi}^j \psi^k , \quad (11)$$

in the gauge $\mathcal{A}_0^a = \psi_0^a = 0$.

The hamiltonian whose vacuum states include solutions to the twisted monopole equations will take the form $H' = \frac{1}{2} \{Q', Q'^\dagger\}$. After some straightforward algebra, one finds the new hamiltonian on the states $\Psi'[A, \psi]$ takes the form

$$H' = 2 \int_Y (F_{0i}^+ - \epsilon_{ijk} \bar{\psi}^j \psi^k)^\dagger (F_{0i}^+ - \epsilon_{imn} \bar{\psi}^m \psi^n) \quad (12)$$

Comparing this hamiltonian to the Floer hamiltonian, we note some interesting differences. First, only the abelian component of the gauge fields play a role. Next, the fermionic partners appear in a fashion which does not preserve ghost number. Note also that none of the fermions have appropriate kinetic contributions. All of these features are consistent with the fact that the twisted monopole equations are simply re-writings of the self-dual curvature condition.

The question remains how to extract the solution Ψ' from the Floer cohomology. That is we seek a W such that given a Floer representative Ψ ,

$$\Psi[\mathcal{A}_i^a, \psi_i^a] = e^{-2\pi t W[\mathcal{A}_i^a, \psi_i^a]} \Psi'[A, \psi] . \quad (13)$$

It is not hard to see that such a functional is given by

$$W[\mathcal{A}_i^a, \psi_i^a] \equiv - \frac{1}{4\pi} \int_Y \text{Tr}(\text{Ad}\mathcal{A} + \frac{2}{3}\mathcal{A}^3) + W'[A, \psi] , \quad (14)$$

Here, the first term in W is recognized as the $SU(2)$ Chern-Simons action. The virtue of the construction (13) is that it allows us to conjecture that given $X \equiv X_l \cup_Y X_r$, where X_l and X_r are manifolds whose boundaries are diffeomorphic to Y but have opposite orientation, the inner products are related by

$$\langle \Psi | \Psi \rangle = \langle \Psi' | \Psi' \rangle . \quad (15)$$

It is reasonable to presume that the Ψ' are representatives of a Floer homology group but for spectral flows governed by the monopole equations and are obtained from the Seiberg-Witten invariants via surgery. In that case, we conjecture that this equality will unlock the formal relationship between the those invariants and the Donaldson polynomials.

4 Other Directions

Apart from the obvious solutions to the usual Floer homology condition, namely Ψ 's which have support only on $\frac{\delta}{\delta \mathcal{A}_i^a(x)} + \frac{1}{2} t \epsilon^{ijk} \mathcal{F}_{jk a}(x) = 0 \leftrightarrow \mathcal{F}_{0i a}^+ = 0$, another simple solution is evident. If the condition

$$\mathcal{F}_{0i a}^+ = \kappa f_{abc} \epsilon_{ijk} \psi^j{}^b \chi^{k c} , \quad (16)$$

is met, then $Q_t = Q_t^\dagger = 0$ for any κ . However, this solutions is not compatible with equation (1) due to the presence of the structure constants and the fact that ψ and χ are canonically conjugate to each other. Beyond this, our methodology in the last section may be extended to construct other cohomologies.

Our procedure suggests another direction to explore. As we discussed before, we do not want to add topological matter to TYM in order to obtain the monopole equations as this would spoil the direct connection with the self-dual curvatures of Donaldson theory. Now, in principle, TYM exists for an arbitrary structure group. With this in mind, we are led to introduce a group which has both bosonic and fermionic generators; i.e., a supergroup. For simplicity, let us take the group to be $OSp(1|2)$ with graded commutators:

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J^c , \\ [J_a, Q_\alpha] &= -i \frac{1}{2} (\gamma_a)_\alpha{}^\beta Q_\beta , \\ \{Q_\alpha, Q_\beta\} &= i (\gamma_a)_{\alpha\beta} J^a . \end{aligned} \quad (17)$$

As these equations suggest, $J_a(Q_\alpha)$ are Grassmann even(odd) generators with $\alpha = 1, 2$ and the J_a forming a $SU(2)$ subgroup. The (γ^a) are three dimensional Clifford matrices: $\gamma^a \in (\sigma^3, -\sigma^1, \sigma^2)$. Introduce a connection one-form on the $OSp(1|2)$ -bundle over X ,

$$\mathbf{A} = \mathcal{A}^a J_a + \Upsilon^\alpha Q_\alpha , \quad (18)$$

where the two component Grassmann odd field Υ^α is (ψ^1, ψ^2) . Its curvature two-form is

$$\begin{aligned} \mathcal{F} &= \hat{F}^a J_a + f^\alpha Q_\alpha , \\ \hat{F}^a &= F^a(\mathcal{A}) J_a + i \frac{1}{2} (\gamma^a)_{\alpha\beta} \Upsilon^\alpha \wedge \Upsilon^\beta J_a , \\ f^\alpha &= D(\mathcal{A}) \Upsilon^\alpha = d\Upsilon^\alpha - \frac{1}{2} \mathcal{A}^a \wedge \Upsilon^\beta (\gamma_a)_\beta{}^\alpha , \end{aligned} \quad (19)$$

where $F(\mathcal{A})$ is the usual curvature of a $SU(2)$ -bundle. It then follows that the self-dual curvature equations become

$$\mathcal{F}^+ = 0 \quad \implies \quad \begin{cases} F^{+a}(\mathcal{A}) = -i \frac{1}{2} (\gamma^a)_{\alpha\beta} P_+(\Upsilon^\alpha \wedge \Upsilon^\beta) , \\ P_+(D(\mathcal{A}) \Upsilon^\alpha) = 0 . \end{cases} \quad (20)$$

By enlarging the principal bundle we have incorporated the monopole equations into a single vanishing self-dual curvature equation.

5 Conclusion

We have realized the twisted version of the monopole equations [1] in the fields space of topological Yang-Mills. In addition, we have identified a unitary transformation between the respective cohomologies and were led to conjecture an equivalence between the Donaldson invariants and those which follow from the spectral flows governed by the monopole equations. Our method suggests a number of generalizations, including the appearance of the monopole equations in a self-dual curvature condition on a super-bundle. In addition, we have found that the twisted monopole equations arise directly from vanishing self-dual curvature condition on an $SU(2)$ -bundle in which the fermions are realized as spanning the tangent space for a specific background. That background being one in which only the connection in the abelian direction is non-zero and its tangent space is null. In this vein, the fermions in the equations span the tangent space to the point zero which is the background value of the connections in the compliment of the $U(1)$ subgroup.

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